Supporting Mathematical Education with Interactive Visual Proofs

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ABSTRACT

While visual proofs are already a valuable tool for conveying the logical ideas behind mathematical proofs, their potential for explaining and engaging can be enhanced by the addition of interactive elements. We present an open platform for interactive visual proofs that is freely available for students and teachers alike at `https://visualproofs.github.io/`, and which can be expanded by the community. Our platform has the potential to support mathematical education by providing students with a more interactive and engaging way to learn mathematical concepts and to show them directly the importance of proofs in mathematics. We aim to support equity in mathematical education by making our platform free, participatory and open to multiple languages. We encourage the community to contribute to its development and expansions through the GitHub repository available under CC BY NC 4.0 International License as well as through conducting future research on the effects of interactive visual proofs in education.

Index Terms: Information systems—World Wide Web—Web applications—Crowdsourcing; Human-Centered Computing—Interaction design—Interaction design process and methods

1 INTRODUCTION

The concept of proofs is considered to be the essential foundation of mathematics. Euclid's *Elements*, dating back at least 2300 years, formalised proofs as the path for establishing mathematical truth, although logical reasoning has been prevalent for even longer. As proof plays a crucial role in mathematical reasoning, teaching proofs is essential to tertiary education in mathematics. However, the importance of proofs in secondary school mathematics has traditionally been limited [10] with many teachers preferring heuristics over proof-based instruction [6]. While proper mathematical verification of theorems may seem complicated to students and their often technical nature can indeed hinder mathematical understanding, there also exist proofs which highlight characterising properties and use them to explain the reason why a certain theorem is valid. These proofs have the potential to aid in improving students' comprehension of mathematical concepts, emphasising the significance of proofs in mathematics, and acclimating students to logical reasoning.

Visual aids have long been used to illustrate proof ideas, with some surviving diagrams dating back to ancient Greece and China [4]. While geometric constructions may be the most apparent example, visual aids are present in all areas of mathematics. Although visual proofs may not be considered valid as proofs by themselves, they serve as powerful pedagogical tools, as they summarise the entire idea of a proof in a single image. Therefore, students only need to comprehend and remember the image, rather than lengthy and complex logical derivations, to understand and reproduce the proof. This method is especially beneficial for visual-pictorial learners [11].

The expansion of digital tools has allowed visual proofs to evolve over the last few decades. Visual proofs may now be presented as videos, which enable the display of motion [13]. Moreover, the use of dynamic geometry software like GeoGebra [9] allows users to interact with mathematical sketches, allowing for the addition of a new form of movement to visual proofs. Unlike a video, these visual proofs are interactive and controlled by the user, adding to their effectiveness as teaching tools. With the continued evolution of the internet and online hosting of such software, it has become possible for creators to share their interactive visual proofs and collect them in an open-source environment.

We have created the first open-source website dedicated to hosting interactive visual proofs. While there exist multiple websites that collect visual proofs [1, 14], they offer only static or animated but not interactive visual proofs. The website Brilliant [3], created by the for-profit company of the same name, offers courses with interactive elements. However, the platform does not primarily focus on proofs but on heuristics. In addition, Brilliant requires a paid subscription to be used in its entirety, which creates a barrier to equitable access, and unlike our platform, there is no direct way for the community to contribute its own content. Our website works towards supporting equity in mathematical education through free access, openness for participation and multilingual options.

2 RELATED WORK

In this section, we will discuss firstly the use of visual proofs in an educational environment and secondly the benefits in the learning experience when adding novel interactive elements to visual proofs.

2.1 Visual proofs as teaching tools

There has been an ongoing discussion in recent years about the inclusion of proofs in secondary education. Some students do not see the value in learning about proofs and see it as a 'self-purpose' that does not contribute towards the goal of promoting mathematical understanding and instead only 'complicate[s an] already difficult situation' [20]. To illustrate this perspective, consider a proof by induction. From the validity of the statement for some natural number n, we infer validity of the same statement for $n+1$ by the induction step. Through proving that the statement holds for some initial natural number, its validity is proven for all following numbers. While this type of proof does in fact *prove* that the statement is true, in most cases it does not *explain* the reason behind its validity and leaves the question of how one first deduced this statement completely nebulous. A corresponding example for the Gauss summation formula is given in Figure 1. From this experience, students may rightfully be confused and discouraged from mathematical rigour, thinking it rather hinders than cultivates mathematical comprehension. Many

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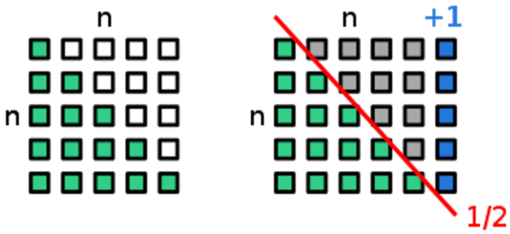

(a) Visual proof

$$P(n+1) = P(n) + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

(b) Induction step in proof by induction

Figure 1: Elements of two proofs for the Gauss summation formula $P(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. The proof by induction may be used to prove the validity of the statement. Unlike the visual proof, the proof by induction does not offer insight to why the statement is true or how one may discover it.

teachers seem to agree with this assessment, discarding the teaching of proofs for a teaching of mathematical heuristics [6].

Clearly, proofs have a vital role in the mathematical framework and therefore cultivating a grasp on mathematical logic is of interest in mathematical education. However, the benefit of teaching proofs extends further than making students comfortable with mathematical reasoning. Proofs fulfil a variety of functions, only one of which is verification [6]. For the classroom, the most important concern is mathematical reasoning. Thus explanatory proofs, which are defined by Hanna as proofs which indicate *why* a certain statement is true, should be given preference over proofs which merely prove, that is proofs which verify the validity of a statement without illuminating the underlying reason for its validity [7]. This functionality of proof is unfortunately often absent from the awareness of teachers [10]. Greater effort needs to be taken by mathematicians and mathematical educators for teachers to recognise the benefit of explanatory proofs.

Visual proofs are an excellent but underused tool to teach students about the reasoning behind a particular proof as well as the concepts behind proofs in general. By design, visual proofs are not using rigorous logic to prove their statement. While this is often seen as a negative and leads to them not being considered as proper proofs for many mathematicians, it means that their contribution is to express the reason why a particular statement is true so exhaustively that an understanding of these ideas suffices to complete a formal proof [16]. Thus, visual proofs are inherently explanatory.

Visual proofs also share the same advantages as other visual aids, which are already frequently used to great effect in teaching mathematics [5, 15]: Firstly, the process of translating a verbal or symbolic mathematical statement into an image already may train an ability to understand and handle mathematical concepts. Geometric constructions, for example, require and cultivate knowledge of the definitions of the underlying objects. Secondly, representing algorithmic steps visually can enhance the comprehension of the algorithm, both as to how it operates and why it fulfils its goal. Thirdly, students are enabled to train the ability to handle mathematical concepts and to think about them in a visual way by translating mathematical, especially non-geometrical, objects such as algebraic expressions into graphical representations (see Figure 2). Research has indeed shown a relation between the ability to process visual proofs and maturity of geometrical thinking [18]. This may indicate that training with visual proofs can support the development of geometric thinking skills, though more research needs to be done to support this claim.

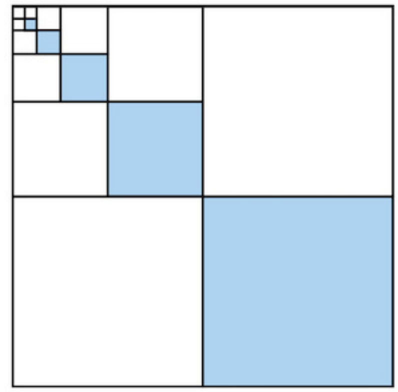

Figure 2: A visual proof for the equality $\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{3}$. Students are taught to visualise the quantity $\frac{1}{4}$ as a fourth of the area of a square and that in a self-similar fashion, $\left(\frac{1}{4}\right)^{n+1}$ can be thought of as a fourth of a square of area $\left(\frac{1}{4}\right)^n$. Through this exercise, students are tasked to envision infinite repetition.

2.2 Interactivity as a compelling feature

The potential for interactivity in visual proofs, which is one of the technical properties that distinguishes our website from printed visual proofs [16] or proofs in form of videos [14], creates various benefits for assisting the learning experience of the user, four of which we outline in this section: Active participation in the learning process; distinguishing chance phenomena from generality in sketches; allowing self-pacing in comprehending steps of a proof or construction; and interactive highlighting to assist focus in complex visuals.

Firstly, interactivity transforms the learning subject from passive consumer to active participant. Active participation through software such as dynamic geometry software has been shown to lead to an improvement in objective criteria such as exam results as well as a more positive self-perception about the confidence in the students' mathematical abilities [2]. Furthermore, interactive software has substantially increased students' enjoyment of the lesson in various studies [2, 22], which is an important part in education.

Secondly, being able to change parameters in visuals, such as with dynamic geometry software like GeoGebra [9], helps to distinguish phenomena that occur by chance from those that hold with generality. This does not only boil down to creating multiple sketches with different parameters without much effort; more so, the use of continuous changes of parameters through sliders better visualises the parameters' gradual effects on other parts of the construction. This helps users to understand the influence of these parameters and aids them in discovering important relationships and constants, which may be an important part of completing a proof. Indeed, successful visual proofs communicate a defining property which the proof depends upon and for generalising from an example of the statement such as an image to a result that holds in general, we need to understand how the theorem responds to variations in the object (see Figure 3) [21]. It is vital to distinguish our usage of added interactivity from the concept of inappropriate generalisation. Students may use such technical implementations to easily check that a result holds in a large variety of cases and therefore incorrectly assume its general validity. This lends support for an idea held by some educators that deductive proofs in geometry should be given less importance or discarded altogether in favour of a completely experimental approach to justifying mathematical concepts. Instead, we aim to use interactive alteration not to merely observe that the result holds under modification of the objects, but to discover which characterising property of these objects, upon which a proof may rely, remains unchanged.

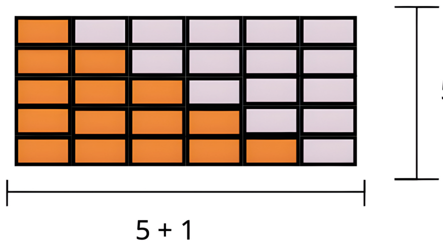

Figure 3: A visual proof for the Gauss summation formula. While this sketch only shows the example $n = 5$, from it we can detect the characterising property that two steps can be stacked in this manner to form a rectangle. We also see that with varying size n of the steps, the size of the rectangle also changes, thus affecting the formula.

Thirdly, the users ability to go through the visual steps of a proof at their own pace and in one graphic instead of multiple images helps to understand long geometric constructions and inductive proofs. This feature provides an active learning experience that engages the user and promotes deeper understanding and retention of the concepts presented.

Lastly, in complicated constructions with a multitude of important parts, interactive highlighting can be used to make sure that the user is not overwhelmed but can concentrate on a proof one part at a time. While the use of a multitude of sketches of the same construction with different highlighting also allows the user to focus on different parts of the proof, interactive highlighting puts the student into an active role and supports them in focusing their attention to where they want it to be. Furthermore, software allows users to zoom into parts of an image, which is especially useful for detailed geometric constructions or induction proofs such as for infinite sums.

3 Design Rationale

We have created the first open-source website dedicated to hosting interactive visual proofs, which can be accessed at `https://visualproofs.github.io/`. The website was generated using Jekyll, Ruby, and Markdown and is currently hosted on GitHub Pages. It consists of two main pages: a homepage and a contribute page. The homepage provides a brief explanation of visual proofs and allows users to freely select proofs that interest them. Through the contribute page, issues or enhancement ideas can be reported through a template and are dealt with through a ticket system mechanism. Code proposals can also be made directly via GitHub Pull Requests. All posts are currently available in trilingual format, in English, German and French, with the option to add additional languages. An example of a post on the Sierpinski triangle is given in Figure 4.

The website offers a wide range of proofs from various mathematical fields, employing different methods. This diversity in content serves to enhance the user's learning experience as it teaches mathematical reasoning skills across multiple domains. The proofs are given tags so that users can easily find other posts on the same topic and deepen their understanding on a certain subject. Furthermore, the presented proofs differ in their inclusion within the mathematical curriculum. While some topics, such as the Pythagorean theorem or polynomial expansion, are commonly taught in secondary mathematical education worldwide, other proofs, such as the pizza theorem, utilise well-known techniques in novel and challenging ways. Finally, certain proofs introduce users to new concepts that they may not have encountered during secondary education, such as fractals. This diversity of content aligns with gamification principles [19, 8]: Students are offered a starting point with familiar concepts to accustom themselves to the structure of the visual proofs and to get into a flow state. The gradual introduction of new challenges and concepts sparks curiosity and surprise. By using familiar concepts in new and creative ways, the website challenges the user to apply their knowledge to novel problems and promotes the development of mathematical intuition. In summary, our website provides a rich and engaging learning experience for users of all backgrounds and skill levels, with the aim of promoting a deeper understanding of mathematical proofs.

The process of proving a mathematical theorem can often be complex and difficult to comprehend, especially for those who are not well-versed in mathematics. However, breaking down the proof into five structured steps may strengthen the understanding and engagement of the reader. These steps are: hook, pattern discovery, generalisation, interactivity and presentation of the mathematical notation.

The first step is the hook, which serves as an introduction to the theorem. It aims to engage the reader by providing context for the theorem and may include information about the mathematician behind the proof or the definition and examples of a new concept related to the theorem.

Next, the theorem is introduced through discovering first patterns. By examining a few simple examples, the reader can better understand the main idea behind the theorem. This step helps to establish a foundation of knowledge that can be built upon as the proof progresses.

Once the main idea has been established, the pattern is then generalised to provide a full visual proof of the theorem. This step helps the reader understand the theorem in a more abstract and general context, expanding their understanding beyond the initial examples.

To increase engagement and promote understanding, the proof includes an element of interactivity. This feature encourages readers to actively engage with the material and can help to increase their understanding and retention of the theorem.

Finally, if the theorem can be expressed by a mathematical formula, it is given at the end. This step serves to recap what has been learned and provides a gentle introduction to mathematical notation. Understanding the theorem in a more formal mathematical context is essential for those who wish to explore the topic further.

By following these five steps, the proof becomes more accessible to a wider audience, promoting a deeper understanding of the theorem. The structured approach helps to break down complex concepts into manageable pieces, making it easier for readers to engage with the material and ultimately increase their knowledge and comprehension.

4 Discussion

As has been widely discussed, visual proofs do not constitute proper proofs in the eyes of the majority of mathematicians. Nevertheless, they bring great benefit for the teaching of proofs in secondary mathematical education, which in our opinion is an integral part in instructing students in mathematical literacy. While visual proofs serve an explanatory purpose in assisting in the comprehension of a proof's content, their larger value lies on a meta-level. These explanatory proofs demonstrate to students how proofs can aid rather than impede their comprehension, instil comfort with the logical principles of proof, and ultimately illustrate the critical functions of proofs in the mathematical framework. Interactive visual proofs, as previously highlighted, aim to offer a more easily understandable and engaging approach to explanatory proofs, yet they are an underutilised pedagogical resource in secondary mathematical education.

We have developed the first platform to host interactive visual proofs in a freely available manner for users and creators alike. Our static website was constructed to propose an integration of interactive

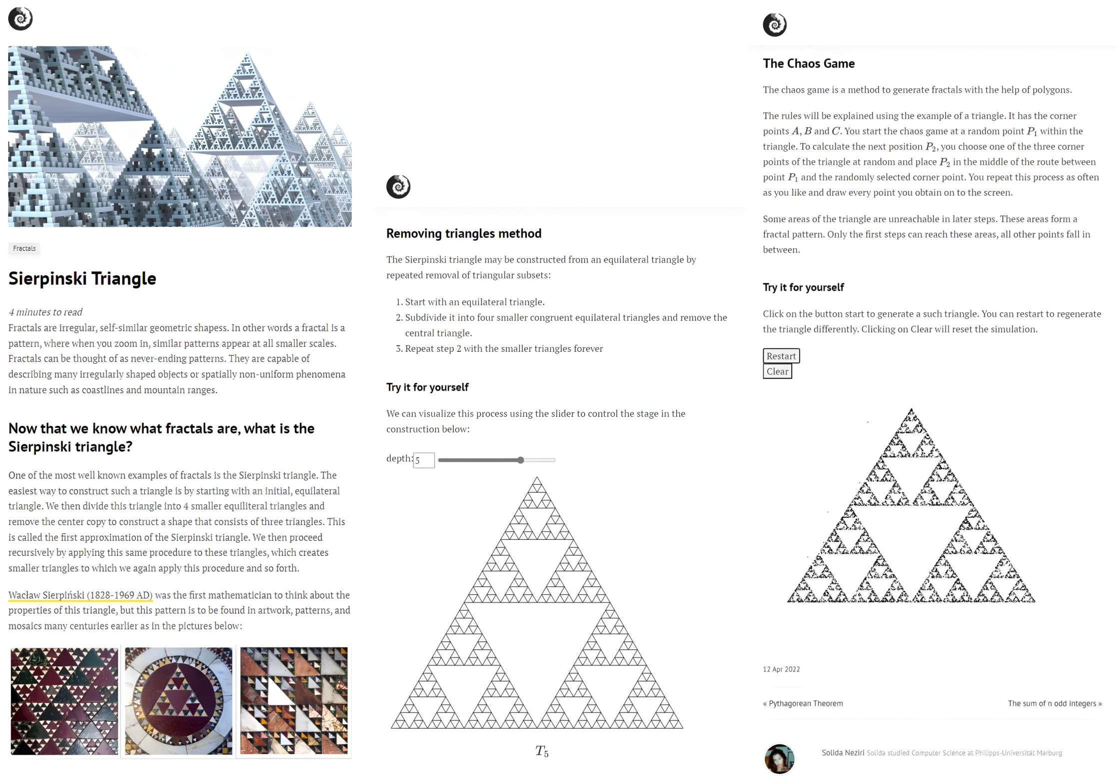

Figure 4: Display of the post on the Sierpinski triangle.

visual proofs into the teaching of diverse mathematical disciplines and allow for equitable access to these new teaching resources. We hope to prompt a dialogue about their educational advantages and how their utilisation may supplement conventional teaching methods. To earnestly evaluate the strengths and limitations of this approach, a theoretical discussion of teaching is insufficient; we must engage with the tools ourselves and, more importantly, observe students using them. User evaluation tailored to interactive visualisation [17] should be undertaken to compare interactive to static visual proofs. Moreover, we believe that our platform, which aggregates these interactive visual proofs and is publicly accessible for both educational content consumption and creation, will permit educators to employ these tools with minimal obstacles. Our intention is for this platform to be utilised in the teaching process and, more critically, to encourage a reevaluation of teaching methodologies and motivate further empirical investigation.

There exist many ways to improve the platform, such as expanding the collection of proofs and translating them into more languages. To further refine the experience of browsing the website, a multi-language plugin could be installed to enable language switching and ensure that users are presented with proofs in their preferred language. Additionally to the plugin which tracks the average reading time, incorporating a plugin to track the average time spent using the interactive elements may improve user engagement. Finally, recognising top contributors through a crowd-sourced scoreboard can improve the contribution aspect of the website.

Our work was predominantly inspired by dynamic geometric software such as GeoGebra, which is utilised to produce interactive sketches displaying mathematical ideas. However, like visual proofs before them, their usage is primarily confined to the mathematical community, where they neatly display concepts already well-understood. Thus, their role is often confined to mathematical entertainment and aesthetic pleasure, severely under-utilising their potential in education. Although more schools now possess modern technology, their use is frequently limited to performing the same tasks as one could with pen and paper. Yet, the development of static sketches does not fully utilise the potential of this technology. That these new gadgets are not exploited to the fullest may also be due in part to the fact that these tools are novel or even completely unknown to most educators. Therefore, it is essential that mathematicians and mathematical educators inform teachers pre-service and in-service about these new teaching tools and create a user-friendly access point to properly leverage these technologies. A further step would be to incorporate interactive features to display the user's changes to the visual proof. This would interactively highlight the corresponding change(s) in the relevant mathematical equation(s).

The advancement of personalised learning has emerged as one of the greatest educational challenge of our time. The United States National Academy of Engineering has recognised it as one of the 14 Grand Challenges for Engineering in the 21$^{\text{st}}$ century and technology-support personalised learning has already been shown to improve learning outcomes [12]. Interactive visual proofs may support this objective, as they provide a more individualised and captivating learning experience for every student at low cost. They support students in attaining a more thorough comprehension of the subject matter and simultaneously demonstrate the importance of proofs in the mathematical context in an illustrative and engaging manner. Amidst uncertainty surrounding the role of proofs in mathematical education and their inclusion in the classroom, interactive visual proofs demonstrate their potential to help, not hinder, our understanding of mathematics.

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